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Mean-field stochastic linear–quadratic optimal control with Markov jump parameters

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ABSTRACT

This paper considers a class of mean-field stochastic linear–quadratic optimal control problems with Markov jump parameters. The new feature of these problems is that means of state and control are incorporated into the systems and the cost functional. Based on the modes of Markov chain, the corresponding decomposition technique of augmented state and control is introduced. It is shown that, under some appropriate conditions, there exists a unique optimal control, which can be explicitly given via solutions of two generalized difference Riccati equations. A numerical example sheds light on the theoretical results established.

1. Introduction

During the past three decades, Markov jump systems have gained a great deal of attention. Such systems often arise in reality with component failures or repairs, changing subsystem interconnections, and abrupting environmental disturbances. It can be found in robotic manipulator systems, aircraft control systems, large scale flexible structures for space stations (such as antenna, solar arrays, among others), and flexible manufacturing systems, on which an actuator or a sensor failure is a quite common occurrence. Without any intention of being exhaustive here, we mention [1–9] and the monographs [10–12] to see different aspects of control problems corresponding to Markov jump systems.

In this paper, a kind of mean-field stochastic linear–quadratic (LQ) optimal control problem with Markov jump parameters is investigated. Compared with the standard stochastic LQ optimal control problems with Markov jump parameters, an important feature of the problem in this paper is that the cost functional involves nonlinearly the states and the controls as well as their expected values. Such a feature roots itself in the category of mean-field theory, which is developed to study the collective behaviors resulting from individuals' mutual interactions in various physical and sociological dynamical systems. There exist many successful applications of the mean-field formulation in various field of engineering, games, finance and economics in the past few years. Recently, stochastic maximum principles of mean-field type are extensively studied in several works [13–15], which specify the necessary conditions for the optimality. As applications, [13,14] studied the Markowitz mean–variance portfolio selection and a class of mean-field LQ problems using stochastic maximum principle. [15] considered mean-field control problems with partial information. [16] investigated the definite mean-field LQ control over a finite time horizon using a variational method and a decoupling technique. It is shown that the optimal control is of linear feedback form and that the gains are represented by solutions of two coupled differential Riccati equations. [17] formulated the discrete-time definite mean-field LQ problem as an operator stochastic LQ optimal control problem. By the kernel-range decomposition representation of the expectation operator and its pseudo-inverse, an optimal control is obtained based on the solutions of two Riccati difference equations. Furthermore, the closed-loop formulation is also investigated. Later, [18,19] generalized results obtained in [16,17] to the case of infinite time horizon.

It is worth noting that the recent research on controlled mean-field stochastic differential and difference equations is partially relighted by a surge of interest in mean-field games [20–27]. Particularly, [21–23] investigated large population stochastic dynamical games with mean-field terms. [24] considered similar

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problems from the viewpoint of mean-field theory. [25,27] dealt with the asymptotically optimal decentralized control problem for the large population multi-agent systems with Markov jump parameters. [26] considered risk-sensitive mean-field games with some interesting aspects, and [28] considered LQ mean-field games via the adjoint equation approach. It is worth pointing out that mean-field games can be reduced to a standard control problem, but the mean-field type control is a non-standard control problem (see [20]). [29,27] studied the mean-field games involving random coefficients. [29] established a stochastic maximum principle for general nonlinear system, which provides necessary conditions for the existence of Nash equilibria in a certain form of N-agent mean-field stochastic differential game. [27] investigated an infinite horizon mean-field LQ games with Markov jump coefficients. Specifically, the distributed strategies were given by solving a Markov jump tracking problem. It is shown that the closed-loop system is uniformly stable, and the distributed strategies are asymptotically optimal in the sense of Nash equilibrium, as the number of agents grows to infinity.

To our knowledge, most of the existing results about mean-field LQ optimal control problems mainly focus on deterministic coefficients. In the real problem, however, one often encounters systems with random coefficients. In the case of deterministic coefficients, it is shown that the optimal controls are linear feedback forms of the state x_k and its expectation $\mathbb{E}x_k$. For a deterministic matrix M_k, $\mathbb{E}(M_kx_k) = M_k\mathbb{E}x_k$, which is an essential property to obtain the optimal control to mean-field LQ problems. But, when M_k becomes random, the property of $\mathbb{E}(M_kx_k) = M_k\mathbb{E}x_k$ no longer holds. This may result in fundamental difficulty in tackling such stochastic control problems with random coefficients (see [16]).

In this paper, we introduce a decomposition technique of the state and the control based on the modes of Markov chain, which is shown to be efficient to attack Markov jump mean-field LQ problem. By completing the square for two different parts of the augmented state and control, the optimal control is constructed via solutions to two generalized difference Riccati equations. The optimal control is shown to be a linear feedback of the current state and its expectation of decomposition of the state.

The decomposition technique adopted in this paper is motivated by [10,3], where a decomposition technique of the state was introduced corresponding to the modes of Markov chain, and the stability of the control-free systems was investigated. In this paper, based on the modes of Markov chain, not only the state and the control are decomposed, but also the mean-field LQ optimal control problem with Markov jump parameters is decomposed to a solvable formulation. By the augmented state and control, we can successfully construct the optimal control of the original mean-field LQ optimal control problem with Markov jumps. A numerical example in Section 4 illustrates that our results are significantly different from those results corresponding to standard Markov jump stochastic LQ problems.

The rest of this paper is organized as follows. Section 2 gives some preliminaries. Section 3 presents the main results of this paper. Section 4 introduces a numerical example. Concluding remarks are given in Section 5.

2. Problem formulation

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space which is assumed to be abundant enough such that two processes $\theta \equiv \{\theta_k\}$, $w \equiv \{w_k\}$ and a random ζ live on it.

(a) θ is a homogeneous Markov chain taking values in a finite set $\{1, .., m\} \equiv \mathcal{M}$ with a stationary one-step transition probability matrix $\Lambda = (p_{ij})$. The (i, j)th entry of Λ is

$$p_{ij} = P(\theta_{k+1} = j|\theta_k = i), \quad i, j \in \mathcal{M},\ k = 0, 1, \ldots. \tag{2.1}$$

The initial distribution of θ_0 is denoted by $\nu = (\nu_1, \ldots, \nu_m)^T$, where the superscript T denotes the transposition of a matrix or a vector.

(b) w is a martingale difference sequence in the sense that $\mathbb{E}[w_{k+1}|\mathcal{F}_k] = 0$ with $\mathcal{F}_k$ being the σ-algebra generated by $\{\zeta, w_l, \theta_l, l = 0, 1, \ldots, k\}$. It is assumed that w has the property

$$\mathbb{E}[(w_{k+1})^2|\mathcal{F}_k] = 1, \tag{2.2}$$

and that θ and w are independent of each other.

(c) ζ is square integrable with a known distribution.

Consider the cost functional

$$\begin{aligned} J(\zeta, u; \theta_0) = &\sum_{k=0}^{N-1} \mathbb{E}\left[x_k^T Q_{\theta_k} x_k + (\mathbb{E}x_k)^T \bar{Q}_{\theta_k} \mathbb{E}x_k \right.\\ &\left. + u_k^T R_{\theta_k} u_k + (\mathbb{E}u_k)^T \bar{R}_{\theta_k} \mathbb{E}u_k\right]\\ &+ \mathbb{E}\left(x_N^T G_{\theta_N} x_N\right) + \mathbb{E}\left[(\mathbb{E}x_N)^T \bar{G}_{\theta_N} \mathbb{E}x_N\right], \end{aligned} \tag{2.3}$$

which is subject to the following dynamics

$$\begin{cases} x_{k+1} = \left[A_{\theta_k}x_k + B_{\theta_k}u_k\right] + \left[C_{\theta_k}x_k + D_{\theta_k}u_k\right]w_k,\\ x_0 = \zeta, \quad k \in \mathbb{T} \equiv \{0, 1, \ldots, N-1\}. \end{cases} \tag{2.4}$$

Here, N is a positive integer; $\{x_k \in \mathbb{R}^n, k \in \bar{\mathbb{T}}\}$ and $\{u_k \in \mathbb{R}^p, k \in \mathbb{T}\}$ are the state process and the control process, respectively, with $\bar{\mathbb{T}} = \{0, 1, \ldots, N\}$; θ represents the mode of system (2.4). When $\theta_k = i \in \mathcal{M}$, $A_{\theta_k}, B_{\theta_k}, C_{\theta_k}, D_{\theta_k}, Q_{\theta_k}, \bar{Q}_{\theta_k}, R_{\theta_k}, \bar{R}_{\theta_k}$ will be denoted by $A^i, B^i, C^i, D^i, Q^i, \bar{Q}^i, R^i, \bar{R}^i$, respectively, which are of compatible dimensions. Similar notations hold for G_{θ_N} and $\bar{G}_{\theta_N}$.

Throughout this paper, θ, w and ζ are assumed to be available to us. Therefore, at time point k, the information set that we have is $\mathcal{F}_{k-1}$. Let $L^2_{\mathcal{F}}(\mathbb{T}; \mathbb{R}^m)$ be the set of $\mathbb{R}^m$-valued processes $\nu = \{\nu_k, k \in \mathbb{T}\}$ such that ν_k is $\mathcal{F}_{k-1}$-measurable and $\sum_{k=0}^{N-1} \mathbb{E}|\nu_k|^2 < \infty$. The optimal control problem of this paper is as follows.

Problem (MF-JLQ). *Given ζ, find a $u^* \in \mathcal{U}_{ad}$ such that*

$$J(\zeta, u^*; \theta_0) = \inf_{u \in L^2_{\mathcal{F}}(\mathbb{T}; \mathbb{R}^m)} J(\zeta, u; \theta_0). \tag{2.5}$$

We call u^* an optimal control for Problem (MF-JLQ).

3. Main results

3.1. System dynamics and cost functional

In [17], the state and the control are decomposed into two orthogonal parts, respectively. By completing the squares for these two parts, we derive the optimal control, which is a linear feedback of the state and its expectation. For a deterministic matrix M_k, we have $\mathbb{E}(M_kx_k) = M_k\mathbb{E}x_k$, which is an essential property to obtain the optimal control [17]. If M_k becomes random, the property $\mathbb{E}(M_kx_k) = M_k\mathbb{E}x_k$ no longer holds. In particular, taking expectation for both sides of (2.4), we have

$$\mathbb{E}x_{k+1} = \mathbb{E}\left[A_{\theta_k}x_k\right] + \mathbb{E}\left[B_{\theta_k}u_k\right].$$

As process θ appears, it is impossible to obtain a deterministic linear system for $\mathbb{E}x_k$. Hence, the results established in [17] cannot be directly applied to solve the case with random coefficients.

To overcome this difficulty, a decomposition technique, corresponding to the modes of Markov chain, is proposed:

$$\begin{cases} y_k^j = x_k I_{(\theta_k=j)}, \quad \forall j \in \mathcal{M},\\ v_k^j = u_k I_{(\theta_k=j)}, \quad \forall j \in \mathcal{M}. \end{cases} \tag{3.1}$$

Based on this decomposition, the optimal control of Problem (MF-JLQ), which gets around the difficulty mentioned above, can be constructed directly.

Simple calculations lead to

$$y^j_{k+1} = \sum_{i=1}^m A^i y^i_k I_{(\theta_{k+1}=j)} + \sum_{i=1}^m B^i v^i_k I_{(\theta_{k+1}=j)} + \left[\sum_{i=1}^m C^i y^i_k I_{(\theta_{k+1}=j)} + \sum_{i=1}^m D^i v^i_k I_{(\theta_{k+1}=j)}\right] w_k, \quad j \in \mathcal{M}. \tag{3.2}$$

To augment the state, let

$$y_k = [y^{1T}_k, \ldots, y^{mT}_k]^T, \qquad v_k = [v^{1T}_k, \ldots, v^{mT}_k]^T, \tag{3.3}$$

$$\mathcal{A} = \begin{bmatrix} A^1 & \cdots & A^m \\ \vdots & & \vdots \\ A^1 & \cdots & A^m \end{bmatrix} \in \mathbb{R}^{nm\times nm}, \quad \mathcal{B} = \begin{bmatrix} B^1 & \cdots & B^m \\ \vdots & & \vdots \\ B^1 & \cdots & B^m \end{bmatrix} \in \mathbb{R}^{nm\times pm},$$

$$\mathcal{C} = \begin{bmatrix} C^1 & \cdots & C^m \\ \vdots & & \vdots \\ C^1 & \cdots & C^m \end{bmatrix} \in \mathbb{R}^{nm\times nm}, \quad \mathcal{D} = \begin{bmatrix} D^1 & \cdots & D^m \\ \vdots & & \vdots \\ D^1 & \cdots & D^m \end{bmatrix} \in \mathbb{R}^{nm\times pm},$$

$$\mathcal{I}^\theta_{k+1} = \begin{bmatrix} I_n I_{(\theta_{k+1}=1)} & 0 & \cdots & 0 \\ 0 & I_n I_{(\theta_{k+1}=2)} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & I_n I_{(\theta_{k+1}=m)} \end{bmatrix} \in \mathbb{R}^{nm\times nm}. \tag{3.4}$$

In the above, $y^{jT}_k = (y^j_k)^T, j \in \mathcal{M}$, and I_n is the identical matrix of n order. Then, from (3.2) we have

$$y_{k+1} = \mathcal{I}^\theta_{k+1}\mathcal{A} y_k + \mathcal{I}^\theta_{k+1}\mathcal{B} v_k + \left[\mathcal{I}^\theta_{k+1}\mathcal{C} y_k + \mathcal{I}^\theta_{k+1}\mathcal{D} v_k\right] w_k. \tag{3.5}$$

On the other hand, taking expectations for both sides of (3.2), we have

$$\mathbb{E}(y^j_{k+1}) = \sum_{i=1}^m A^i \mathbb{E}[y^i_k I_{(\theta_{k+1}=j)}] + \sum_{i=1}^m \bar{B}^i \mathbb{E} v^i_k \mathbb{E}(I_{(\theta_{k+1}=j)}).$$

Since

$$\mathbb{E}[y^i_k I_{(\theta_{k+1}=j)}] = \mathbb{E}\big[\mathbb{E}[y^i_k I_{(\theta_{k+1}=j)}|\mathcal{F}_k]\big] = \mathbb{E}\big[\mathbb{E}[I_{(\theta_{k+1}=j)}|\mathcal{F}_k] y^i_k\big] = p_{ij}\mathbb{E} y^i_k,$$

we have

$$\mathbb{E}(y^j_{k+1}) = \sum_{i=1}^m p_{ij} A^i \mathbb{E} y^i_k + \sum_{i=1}^m p_{ij} B^i \mathbb{E} v^i_k. \tag{3.6}$$

Denote

$$\bar{\mathcal{A}} = \begin{bmatrix} p_{11}A^1 & \cdots & p_{m1}A^m \\ \vdots & & \vdots \\ p_{1m}A^1 & \cdots & p_{mm}A^m \end{bmatrix} \in \mathbb{R}^{nm\times nm}, \quad \bar{\mathcal{B}} = \begin{bmatrix} p_{11}B^1 & \cdots & p_{m1}B^m \\ \vdots & & \vdots \\ p_{1m}B^1 & \cdots & p_{mm}B^m \end{bmatrix} \in \mathbb{R}^{nm\times pm}.$$

Then, we get

$$\mathbb{E} y_{k+1} = \bar{\mathcal{A}}\mathbb{E} y_k + \bar{\mathcal{B}}\mathbb{E} v_k, \tag{3.7}$$

and

$$\mathbb{E}[x^T_k Q_{\theta_k} x_k] = \sum_{i=1}^m \mathbb{E}[x^T_k Q^i x_k I_{(\theta_k=i)}] = \sum_{i=1}^m \mathbb{E}[y^{iT}_k Q^i y^i_k],$$

$$\mathbb{E}\big[(\mathbb{E}x_k)^T \bar{Q}_{\theta_k}\mathbb{E}x_k\big] = \sum_{i,j=1}^m (\mathbb{E}y^i_k)^T \bar{Q}_k \mathbb{E} y^j_k.$$

Here,

$$\bar{Q}_k = \mathbb{E}\bar{Q}_{\theta_k} = \bar{p}^1_k \bar{Q}^1 + \cdots + \bar{p}^m_k \bar{Q}^m$$

with $\bar{p}^j_k = P(\theta_k = j), j \in \mathcal{M}$. Similar equalities hold for $\mathbb{E}[u^T_k R_{\theta_k} u_k]$ and $\mathbb{E}\big[(\mathbb{E}u_k)^T \bar{R}_{\theta_k}\mathbb{E}u_k\big]$. Therefore, $J(x_0, u; \theta_0)$ can be represented as

$$J(x_0, u; \theta_0) = \sum_{k=0}^{N-1}\Bigg\{\sum_{i=1}^m \mathbb{E}[y^{iT}_k Q^i y^i_k] + \sum_{i,j=1}^m (\mathbb{E}y^i_k)^T \bar{Q}_k \mathbb{E} y^j_k + \sum_{i=1}^m \mathbb{E}[v^{iT}_k R^i v^i_k] + \sum_{i,j=1}^m (\mathbb{E}v^i_k)^T \bar{R}_k \mathbb{E} v^j_k\Bigg] + \sum_{i=1}^m \mathbb{E}[y^{iT}_N G^i y^i_N] + \sum_{i,j=1}^m (\mathbb{E}y^i_N)^T \bar{G}_k \mathbb{E} y^j_N, \tag{3.8}$$

where

$$\bar{R}_k = \mathbb{E}\bar{R}_{\theta_k} = \bar{p}^1_k \bar{R}^1 + \cdots + \bar{p}^m_k \bar{R}^m,$$
$$\bar{G} = \mathbb{E}\bar{G}_{\theta_N} = \bar{p}^1_N \bar{G}^1 + \cdots + \bar{p}^m_N \bar{G}^m.$$

Let

$$\mathcal{Q}^1 = \mathrm{diag}\big\{Q^1, \ldots, Q^m\big\} \in \mathbb{R}^{nm\times nm},$$

$$\mathcal{Q}^2_k = \begin{bmatrix} \bar{Q}_k & \bar{Q}_k & \cdots & \bar{Q}_k \\ \bar{Q}_k & \bar{Q}_k & \cdots & \bar{Q}_k \\ \vdots & \vdots & & \vdots \\ \bar{Q}_k & \bar{Q}_k & \cdots & \bar{Q}_k \end{bmatrix} \in \mathbb{R}^{nm\times nm},$$

$$\mathcal{R}^1 = \mathrm{diag}\big\{R^1, \ldots, R^m\big\} \in \mathbb{R}^{pm\times pm},$$

$$\mathcal{R}^2_k = \begin{bmatrix} \bar{R}_k & \bar{R}_k & \cdots & \bar{R}_k \\ \bar{R}_k & \bar{R}_k & \cdots & \bar{R}_k \\ \vdots & \vdots & & \vdots \\ \bar{R}_k & \bar{R}_k & \cdots & \bar{R}_k \end{bmatrix} \in \mathbb{R}^{pm\times pm},$$

$$\mathcal{G}^1 = \mathrm{diag}\big\{G^1, \ldots, G^m\big\} \in \mathbb{R}^{pn\times pn},$$

$$\mathcal{G}^2 = \begin{bmatrix} \bar{G} & \bar{G} & \cdots & \bar{G} \\ \bar{G} & \bar{G} & \cdots & \bar{G} \\ \vdots & \vdots & & \vdots \\ \bar{G} & \bar{G} & \cdots & \bar{G} \end{bmatrix} \in \mathbb{R}^{nm\times nm}.$$

Then, by (3.8), we have

$$J(x_0, u; \theta_0) = \sum_{k=0}^{N-1}\mathbb{E}\Big\{y^T_k \mathcal{Q}^1 y_k + (\mathbb{E}y_k)^T \mathcal{Q}^2_k \mathbb{E} y_k + v^T_k \mathcal{R}^1 v_k + (\mathbb{E}v_k)^T \mathcal{R}^2_k \mathbb{E} v_k\Big\} + \mathbb{E}\big[y^T_N \mathcal{G}^1 y_N\big] + (\mathbb{E}y_N)^T \mathcal{G}^2 \mathbb{E} y_N, \equiv \mathcal{J}(y_0, v; \theta_0). \tag{3.9}$$

3.2. Quadratic dynamics

We now introduce two sets of block-diagonal symmetric matrices $\{\mathcal{P}_k, k \in \bar{N}\}$ and $\{\mathcal{S}_k, k \in \bar{N}\}$ with $\mathcal{P}_k, \mathcal{S}_k \in \mathbb{R}^{nm\times nm}$, which evoke the dynamics (called quadratic dynamics) of $\mathbb{E}\big[y_{k+1}^T\mathcal{P}_{k+1}y_{k+1}\big]$ and $(\mathbb{E}y_{k+1})^T\mathcal{S}_{k+1}\mathbb{E}y_{k+1}$. Note that

$$\mathbb{E}\big[y_N^T\mathcal{P}_Ny_N\big] = \mathbb{E}\big[y_0^T\mathcal{P}_0y_0\big] + \sum_{k=0}^{N-1}\Big\{\mathbb{E}\big[y_{k+1}^T\mathcal{P}_{k+1}y_{k+1}\big] - \mathbb{E}\big[y_k^T\mathcal{P}_ky_k\big]\Big\}$$

and

$$\mathcal{A}^T\mathcal{I}_{k+1}^\theta\mathcal{P}_{k+1}\mathcal{I}_{k+1}^\theta\mathcal{A} = \begin{bmatrix} A^{1T}\left(\sum_{i=1}^m P_{k+1}^{ii}I_{(\theta_{k+1}=i)}\right)A^1 & \cdots & A^{1T}\left(\sum_{i=1}^m P_{k+1}^{ii}I_{(\theta_{k+1}=i)}\right)A^m \\ \vdots & & \vdots \\ A^{mT}\left(\sum_{i=1}^m P_{k+1}^{ii}I_{(\theta_{k+1}=i)}\right)A^1 & \cdots & A^{mT}\left(\sum_{i=1}^m P_{k+1}^{ii}I_{(\theta_{k+1}=i)}\right)A^m \end{bmatrix}$$

with $P_{k+1}^{ii} \in \mathbb{R}^{n\times n}$ being the (i, i)th block of $\mathcal{P}_{k+1}$, $i = 1, \ldots, m$. Then, we have

$$\begin{aligned} &\mathbb{E}\big[y_k^T\mathcal{A}^T\mathcal{I}_{k+1}^\theta\mathcal{P}_{k+1}\mathcal{I}_{k+1}^\theta\mathcal{A}y_k\big] \\ &= \sum_{i_1,i_2=1}^m \mathbb{E}\left[y_k^{i_1T}A^{i_1T}\left(\sum_{i=1}^m P_{k+1}^{ii}I_{(\theta_{k+1}=i)}\right)A^{i_2}y_k^{i_2}\right] \\ &= \sum_{i_1,i_2=1}^m Tr\left[\mathbb{E}\left(A^{i_1T}\left(\sum_{i=1}^m P_{k+1}^{ii}I_{(\theta_{k+1}=i)}\right)A^{i_2}y_k^{i_2}y_k^{i_1T}\right)\right] \\ &= \sum_{l=1}^m Tr\left[\mathbb{E}\left(A^{lT}\left(\sum_{i=1}^m P_{k+1}^{ii}I_{(\theta_{k+1}=i)}\right)A^ly_k^ly_k^{lT}\right)\right] \\ &= \sum_{l=1}^m \mathbb{E}\left(y_k^{lT}A^{lT}\left(\sum_{i=1}^m P_{k+1}^{ii}p_{li}\right)A^ly_k^l\right), \end{aligned} \tag{3.10}$$

where the third equality is by the fact $y_k^{i_2}y_k^{i_1T} = x_kI_{(\theta_k=i_2)}x_k^TI_{(\theta_k=i_1)} = 0$, $i_1 \neq i_2$, and the fourth equality is by the Markovian property of (θ, x). Note that

$$\begin{aligned} &\mathrm{diag}\left\{A^{1T}\left(\sum_{i=1}^m P_{k+1}^{ii}p_{1i}\right)A^1, \ldots, A^{mT}\left(\sum_{i=1}^m P_{k+1}^{ii}p_{mi}\right)A^m\right\} \\ &= \mathrm{diag}\{A^{1T}, \ldots, A^{mT}\}\cdot \mathrm{diag}\Big\{\sum_{i=1}^m P_{k+1}^{ii}p_{1i}, \ldots, \sum_{i=1}^m P_{k+1}^{ii}p_{mi}\Big\} \\ &\quad \cdot \mathrm{diag}\{A^1, \ldots, A^m\} \\ &\equiv \mathcal{A}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{A}^0 \end{aligned}$$

with

$$\mathcal{A}^0 \equiv \mathrm{diag}\{A^1, \ldots, A^m\},$$

$$\pi(\mathcal{P}_{k+1}) \equiv \mathrm{diag}\Big\{\sum_{i=1}^m P_{k+1}^{ii}p_{1i}, \ldots, \sum_{i=1}^m P_{k+1}^{ii}p_{mi}\Big\}.$$

Then, by (3.10), we get

$$\mathbb{E}\big[y_k^T\mathcal{A}^T\mathcal{I}_{k+1}^\theta\mathcal{P}_{k+1}\mathcal{I}_{k+1}^\theta\mathcal{A}y_k\big] = \mathbb{E}\big[y_k^T\mathcal{A}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{A}^0y_k\big]. \tag{3.11}$$

Similarly, we have

$$\begin{cases} \mathbb{E}\big[v_k^T\mathcal{B}^T\mathcal{I}_{k+1}^\theta\mathcal{P}_{k+1}\mathcal{I}_{k+1}^\theta\mathcal{B}v_k\big] = \mathbb{E}\big[v_k^T\mathcal{B}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{B}^0v_k\big], \\ \mathbb{E}\big[y_k^T\mathcal{A}^T\mathcal{I}_{k+1}^\theta\mathcal{P}_{k+1}\mathcal{I}_{k+1}^\theta\mathcal{B}v_k\big] = \mathbb{E}\big[y_k^T\mathcal{A}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{B}^0v_k\big], \\ \mathbb{E}\big[y_k^T\mathcal{C}^T\mathcal{I}_{k+1}^\theta\mathcal{P}_{k+1}\mathcal{I}_{k+1}^\theta\mathcal{C}y_k\big] = \mathbb{E}\big[y_k^T\mathcal{C}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{C}^0y_k\big], \\ \mathbb{E}\big[v_k^T\mathcal{D}^T\mathcal{I}_{k+1}^\theta\mathcal{P}_{k+1}\mathcal{I}_{k+1}^\theta\mathcal{D}v_k\big] = \mathbb{E}\big[v_k^T\mathcal{D}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{D}^0v_k\big], \\ \mathbb{E}\big[y_k^T\mathcal{C}^T\mathcal{I}_{k+1}^\theta\mathcal{P}_{k+1}\mathcal{I}_{k+1}^\theta\mathcal{D}v_k\big] = \mathbb{E}\big[y_k^T\mathcal{C}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{D}^0v_k\big], \end{cases} \tag{3.12}$$

where

$$\mathcal{B}^0 = \mathrm{diag}\{B^1, \ldots, B^m\},$$
$$\mathcal{C}^0 = \mathrm{diag}\{C^1, \ldots, C^m\},$$
$$\mathcal{D}^0 = \mathrm{diag}\{D^1, \ldots, D^m\}.$$

It follows from (3.5), (3.11) and (3.12) that

$$\begin{aligned} &\mathbb{E}\big[y_N^T\mathcal{P}_Ny_N\big] - \mathbb{E}\big[y_0^T\mathcal{P}_0y_0\big] \\ &= \sum_{k=0}^{N-1}\Big\{\mathbb{E}\big[y_k^T\big(\mathcal{A}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{A}^0 + \mathcal{C}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{C}^0 - \mathcal{P}_k\big)y_k\big] \\ &\quad + 2\mathbb{E}\big[y_k^T\big(\mathcal{A}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{B}^0 + \mathcal{C}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{D}^0\big)v_k\big] \\ &\quad + \mathbb{E}\big[v_k^T\big(\mathcal{B}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{B}^0 + \mathcal{D}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{D}^0\big)v_k\big]\Big\}, \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} &(\mathbb{E}y_N)^T\mathcal{S}_N\mathbb{E}y_N - (\mathbb{E}y_0)^T\mathcal{S}_0\mathbb{E}y_0 \\ &= \sum_{k=0}^{N-1}\Big\{(\mathbb{E}y_{k+1})^T\mathcal{S}_{k+1}\mathbb{E}y_{k+1} - (\mathbb{E}y_k)^T\mathcal{S}_k\mathbb{E}y_k\Big\} \\ &= \sum_{k=0}^{N-1}\Big\{(\mathbb{E}y_k)^T\big(\bar{\mathcal{A}}^T\mathcal{S}_{k+1}\bar{\mathcal{A}} - \mathcal{S}_k\big)\mathbb{E}y_k + 2(\mathbb{E}y_k)^T\bar{\mathcal{A}}^T\mathcal{S}_{k+1}\bar{\mathcal{B}}\mathbb{E}v_k \\ &\quad + (\mathbb{E}v_k)^T\bar{\mathcal{B}}^T\mathcal{S}_{k+1}\bar{\mathcal{B}}\mathbb{E}v_k\Big\}. \end{aligned} \tag{3.14}$$

3.3. Optimal control

By (3.13) and (3.14), we have

$$\begin{aligned} &\mathcal{J}(y_0, v; \theta_0) \\ &= \mathbb{E}\big[y_0^T\mathcal{P}_0y_0\big] + (\mathbb{E}y_0)^T\mathcal{S}_0\mathbb{E}y_0 \\ &\quad + \sum_{k=0}^{N-1}\Big\{\mathbb{E}\big[y_k^T\big(\mathcal{Q}^1 + \mathcal{A}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{A}^0 \\ &\quad + \mathcal{C}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{C}^0 - \mathcal{P}_k\big)y_k\big] + 2\mathbb{E}\big[y_k^T\big(\mathcal{A}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{B}^0 \\ &\quad + \mathcal{C}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{D}^0\big)v_k\big] + \mathbb{E}\big[v_k^T\big(\mathcal{R}^1 + \mathcal{B}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{B}^0 \\ &\quad + \mathcal{D}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{D}^0\big)v_k\big] + (\mathbb{E}y_k)^T \\ &\quad \times \big(\mathcal{Q}_k^2 + \bar{\mathcal{A}}^T\mathcal{S}_{k+1}\bar{\mathcal{A}} - \mathcal{S}_k\big)\mathbb{E}y_k + 2(\mathbb{E}y_k)^T\bar{\mathcal{A}}^T\mathcal{S}_{k+1}\bar{\mathcal{B}}\mathbb{E}v_k \\ &\quad + (\mathbb{E}v_k)^T\big(\mathcal{R}_k^2 + \bar{\mathcal{B}}^T\mathcal{S}_{k+1}\bar{\mathcal{B}}\big)\mathbb{E}v_k\Big\} \\ &\quad + (\mathbb{E}y_N)^T\big(\mathcal{G}^2 - \mathcal{S}_N\big)\mathbb{E}y_N + \mathbb{E}\big[y_N^T(\mathcal{G}^1 - \mathcal{P}_N)y_N\big]. \end{aligned} \tag{3.15}$$

To follow the method of completing the square, we now adopt the coordinate $\{(y_k - \mathcal{I}_k^\theta\mathbb{E}y_k, \mathbb{E}y_k, v_k - \mathcal{I}_k^\theta\mathbb{E}v_k, \mathbb{E}v_k), k \in \mathbb{T}\}$, where $\mathcal{I}_k^\theta$ is defined in (3.4). In the following, we shall show that $v_k - \mathcal{I}_k^\theta\mathbb{E}v_k$ and $\mathbb{E}v_k$ can be separately designed, and thus, v_k is obtained. Simple calculations yield

$$\begin{aligned} &\mathbb{E}\big[(y_k^i - I_{(\theta_k=i)}\mathbb{E}y_k^i)^TQ^i(y_k^i - I_{(\theta_k=i)}\mathbb{E}y_k^i)\big] \\ &= \mathbb{E}\big[y_k^{iT}Q^iy_k^i\big] - (\mathbb{E}y_k^i)^T\big[(2 - \bar{p}_k^i)Q^i\big]\mathbb{E}y_k^i, \end{aligned}$$

and

$$\begin{aligned} (\mathbb{E}y_k)^T\mathcal{Q}^1\mathbb{E}y_k &= \mathbb{E}\big[\big(y_k - \mathcal{I}_k^\theta\mathbb{E}y_k\big)^T\mathcal{Q}^1\big(y_k - \mathcal{I}_k^\theta\mathbb{E}y_k\big)\big] \\ &\quad + (\mathbb{E}y_k)^T(2I_{mn} - \mathbb{P}_k)\mathcal{Q}^1\mathbb{E}y_k, \end{aligned}$$

where

$$\mathbb{P}_k = \mathrm{diag}\{I_n\bar{p}_k^1, \ldots, I_n\bar{p}_k^m\}. \tag{3.16}$$

Furthermore, from (3.15) we have

$$
\begin{aligned}
&\mathcal{J}(y_0, v; \theta_0)\\
&= \mathbb{E}\big[y_0^T \mathcal{P}_0 y_0\big] + (\mathbb{E}y_0)^T \mathcal{S}_0 \mathbb{E}y_0\\
&+ \sum_{k=0}^{N-1}\Big\{\mathbb{E}\big[(y_k - \mathcal{I}_k^\theta \mathbb{E}y_k)^T\big(\mathcal{Q}^1 + \mathcal{A}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{A}^0\\
&+ \mathcal{C}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{C}^0 - \mathcal{P}_k\big)(y_k - \mathcal{I}_k^\theta \mathbb{E}y_k)\big]\\
&+ 2\mathbb{E}\big[(y_k - \mathcal{I}_k^\theta \mathbb{E}y_k)^T\\
&\times \big(\mathcal{A}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{B}^0 + \mathcal{C}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{D}^0\big)(v_k - \mathcal{I}_k^\theta \mathbb{E}v_k)\big]\\
&+ \mathbb{E}\big[(v_k - \mathcal{I}_k^\theta \mathbb{E}v_k)^T\big(\mathcal{R}^1 + \mathcal{B}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{B}^0\\
&+ \mathcal{D}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{D}^0\big)(v_k - \mathcal{I}_k^\theta \mathbb{E}v_k)\big]\Big\}\\
&+ \sum_{k=0}^{N-1}\Big\{(\mathbb{E}y_k)^T\big[\mathcal{Q}_k^2 + \bar{\mathcal{A}}^T \mathcal{S}_{k+1}\bar{\mathcal{A}} - \mathcal{S}_k + (2I_{mn} - \mathbb{P}_k)\\
&\times \big(\mathcal{Q}^1 + \mathcal{A}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{A}^0 + \mathcal{C}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{C}^0 - \mathcal{P}_k\big)\big]\mathbb{E}y_k\\
&+ 2(\mathbb{E}y_k)^T\big[\bar{\mathcal{A}}^T \mathcal{S}_{k+1}\bar{\mathcal{B}} + (2I_{mn} - \mathbb{P}_k)\big(\mathcal{A}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{B}^0\\
&+ \mathcal{C}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{D}^0\big)\big]\mathbb{E}v_k + (\mathbb{E}v_k)^T\big[\mathcal{R}_k^2 + \bar{\mathcal{B}}^T \mathcal{S}_{k+1}\bar{\mathcal{B}}\\
&+ (2I_{mn} - \mathbb{P}_k)\big(\mathcal{R}^1 + \mathcal{B}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{B}^0\\
&+ \mathcal{D}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{D}^0\big)\big]\mathbb{E}v_k\Big\}\\
&+ (\mathbb{E}y_N)^T\big(\mathcal{G}^2 - \mathcal{S}_N + (2I_{mn} - \mathbb{P}_N)(\mathcal{G}^1 - \mathcal{P}_N)\big)\mathbb{E}y_N\\
&+ \mathbb{E}\big[(y_N - \mathcal{I}_N^\theta \mathbb{E}y_N)^T(\mathcal{G}^1 - \mathcal{P}_N)(y_N - \mathcal{I}_N^\theta \mathbb{E}y_N)\big]. \qquad (3.17)
\end{aligned}
$$

To proceed, we introduce two difference Riccati equations (DREs). One is

$$
\begin{cases}
\mathcal{P}_k = \mathcal{Q}^1 + \mathcal{A}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{A}^0 + \mathcal{C}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{C}^0\\
\qquad - \mathcal{H}_k^{1T}(\mathcal{W}_k^1)^{-1}\mathcal{H}_k^1,\\
\mathcal{P}_N = \mathcal{G}^1, \quad k \in \mathbb{T}
\end{cases} \tag{3.18}
$$

with

$$
\begin{cases}
\mathcal{W}_k^1 = \mathcal{R}^1 + \mathcal{B}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{B}^0 + \mathcal{D}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{D}^0,\\
\mathcal{H}_k^1 = \mathcal{B}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{A}^0 + \mathcal{D}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{C}^0.
\end{cases} \tag{3.19}
$$

The other one is

$$
\begin{cases}
\mathcal{S}_k = \big[\mathcal{Q}_k^2 + (2I_{mn} - \mathbb{P}_k)\big(\mathcal{Q}^1 + \mathcal{A}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{A}^0\\
\quad + \mathcal{C}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{C}^0 - \mathcal{P}_k\big)\big] + \bar{\mathcal{A}}^T \mathcal{S}_{k+1}\bar{\mathcal{A}} - \mathcal{H}_k^{2T}(\mathcal{W}_k^2)^{-1}\mathcal{H}_k^2,\\
\mathcal{S}_N = \mathcal{G}^2, \quad k \in \mathbb{T}
\end{cases} \tag{3.20}
$$

with

$$
\begin{cases}
\mathcal{W}_k^2 = \big[\mathcal{R}_k^2 + (2I_{mn} - \mathbb{P}_k)\big(\mathcal{R}^1 + \mathcal{B}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{B}^0\\
\quad + \mathcal{D}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{D}^0\big)\big] + \bar{\mathcal{B}}^T \mathcal{S}_{k+1}\bar{\mathcal{B}},\\
\mathcal{H}_k^2 = \bar{\mathcal{B}}^T \mathcal{S}_{k+1}\bar{\mathcal{A}} + (2I_{mn} - \mathbb{P}_k)\big(\mathcal{B}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{A}^0\\
\quad + \mathcal{D}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{C}^0\big).
\end{cases} \tag{3.21}
$$

The following lemma gives the solvability of (3.18) and (3.20).

Lemma 3.1. *If $Q^i, \bar{Q}^i, G^i, \bar{G}^i \ge 0, R^i, \bar{R}^i > 0, i \in \mathcal{M}$, then the DREs (3.18) and (3.20) are solvable.*

Proof. The DRE (3.18) can be rewritten in elements as

$$
\begin{cases}
P_k^j = Q^j + A^{jT}\pi_j(\mathcal{P}_{k+1})A^j + C^{jT}\pi_j(\mathcal{P}_{k+1})C^j\\
\qquad - H_k^{jT}(W_k^j)^{-1}H_k^j,\\
P_N^j = G_N^j, \quad k \in \mathbb{T}, \ j \in \mathcal{M},
\end{cases} \tag{3.22}
$$

where

$$
\begin{cases}
\pi_j(\mathcal{P}_{k+1}) = \sum_{i=1}^m p_{ji}P_{k+1}^i\\
W_k^j = R^j + B^{jT}\pi_j(\mathcal{P}_{k+1})B^j + D^{jT}\pi_j(\mathcal{P}_{k+1})D^j,\\
H_k^j = B^{jT}\pi_j(\mathcal{P}_{k+1})A^j + D^{jT}\pi_j(\mathcal{P}_{k+1})C^j.
\end{cases} \tag{3.23}
$$

These equations are versions of coupled Riccati equations of standard Markov jump LQ problems [10], and thus, are solvable with property $P_k^j \ge 0, k \in \mathbb{T}, j \in \mathcal{S}$. In fact, we can consider the following Markov jump LQ problem for any given initial pair $(x_k, j) \in \mathbb{R}^n \times \mathcal{M}$:

$$
\begin{cases}
\text{minimize } J(x_k, u; j)\\
\quad = \mathbb{E}\left[\sum_{l=k}^{N-1}\big(x_l^T Q_{\theta_l}x_l + u_l^T R_{\theta_l}u_l\big) + x_N^T G_{\theta_N}x_N \,\Big|\, \theta_k = j\right],\\
\text{subject to } x_{l+1} = \big[A_{\theta_l}x_l + B_{\theta_l}u_l\big] + \big[C_{\theta_l}x_l + D_{\theta_l}u_l\big]w_l,\\
\quad l = k, \ldots, N-1.
\end{cases}
$$

Simply completing the square, it holds that

$$
\inf_u J(x_k, u; j) = \mathbb{E}\big[x_k^T P_k^j x_k\big] \ge 0,
$$

where the inequality holds because $J(x_k, u; j) \ge 0$ for any u. Hence, we have $P_k^j \ge 0$ and (3.18) is solvable. Moreover, as

$$
\begin{cases}
2I_{mn} - \mathbb{P}_k > 0,\\
\mathcal{Q}_k^2 + (2I_{mn} - \mathbb{P}_k)\big(\mathcal{Q}^1 + \mathcal{A}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{A}^0\\
\quad + \mathcal{C}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{C}^0 - \mathcal{P}_k\big)\\
\quad = \mathcal{Q}_k^2 + (2I_{mn} - \mathbb{P}_k)\mathcal{H}_k^{1T}(\mathcal{W}_k^1)^{-1}\mathcal{H}_k^1 \ge 0,\\
\mathcal{R}_k^2 + (2I_{mn} - \mathbb{P}_k)\big(\mathcal{R}^1 + \mathcal{B}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{B}^0\\
\quad + \mathcal{D}^{0T}\pi(\mathcal{P}_{k+1})\mathcal{D}^0\big) > 0,
\end{cases}
$$

the DRE (3.20) is a standard Riccati equation, and thus, is solvable with property $\mathcal{S}_k \ge 0, k \in \mathbb{T}$. □

We now state the main result of this paper.

Theorem 3.1. *If $Q^i, \bar{Q}^i, G^i, \bar{G}^i \ge 0, R^i, \bar{R}^i > 0, i \in \mathcal{M}$, then the optimal control of Problem (MF-JLQ) uniquely exists and can be expressed as follows:*

$$
\begin{aligned}
u_k^* &= -(W_{\theta_k})^{-1}H_{\theta_k}x_k + \sum_{j=1}^m (W_k^j)^{-1}H_k^j I_{(\theta_k=j)}\mathbb{E}y_k^j\\
&\quad - \mathcal{I}_m \mathcal{I}_k^\theta(\mathcal{W}_k^2)^{-1}\mathcal{H}_k^2\mathbb{E}y_k, \quad k \in \mathbb{T}.
\end{aligned} \tag{3.24}
$$

Here, $W_{\theta_k} = W_k^j, H_{\theta_k} = H_k^j$ with W_k^j, H_k^j defined in (3.23) when $\theta_k = j$; $\mathcal{W}_k^2, \mathcal{H}_k^2$ are defined in (3.21); $\mathcal{I}_k^\theta$ are defined in (3.4); y_k is defined in (3.3); and

$$
\mathcal{I}_m = (I_n, \ldots, I_n)_{n\times mn}
$$

with I_n being the identical matrix of n order. Under (3.24), the value function is given by

$$
V(x_0, \theta_0) = \inf_{u\in\mathcal{U}_{ad}} J(x_0, u; \theta_0) = \mathbb{E}\big[y_0^T \mathcal{P}_0 y_0\big] + (\mathbb{E}y_0)^T \mathcal{S}_0 \mathbb{E}y_0,
$$

where $\{\mathcal{P}_k, k \in \bar{\mathbb{T}}\}$ and $\{\mathcal{S}_k, k \in \bar{\mathbb{T}}\}$ are defined in (3.18) and (3.20), respectively.

Proof. Under the condition $Q^i, \bar{Q}^i, G^i, \bar{G}^i \ge 0, R^i, \bar{R}^i > 0, i \in \mathcal{M}$, there exists a $c > 0$ such that

$$
J(x_0, u; \theta_0) \ge c\sum_{k=0}^{N-1}\mathbb{E}|u_k|^2.
$$

Hence, $J(x_0, u; \theta_0)$ is coercively quadratic with respect to u. Therefore, the optimal control of Problem (MF-JLQ) exists uniquely. By Lemma 3.1 and completing the square, we have

$$\begin{aligned} J(x_0, u; \theta_0) &= \mathcal{J}(y_0, v; \theta_0) \\ &= \mathbb{E}\big[y_0^T \mathcal{P}_0 y_0\big] + (\mathbb{E}y_0)^T \mathcal{S}_0 \mathbb{E}y_0 \\ &\quad + \sum_{k=0}^{N-1} \mathbb{E}\Big\{\big[v_k - \mathcal{I}_k^\theta \mathbb{E}v_k + (\mathcal{W}_k^1)^{-1}\mathcal{H}_k^1(y_k - \mathcal{I}_k^\theta \mathbb{E}y_k)\big]^T \\ &\quad \times \mathcal{W}_k^1\big[v_k - \mathcal{I}_k^\theta \mathbb{E}v_k + (\mathcal{W}_k^1)^{-1}\mathcal{H}_k^1(y_k - \mathcal{I}_k^\theta \mathbb{E}y_k)\big]\Big\} \\ &\quad + \sum_{k=0}^{N-1} \mathbb{E}\Big\{\big[\mathbb{E}v_k + (\mathcal{W}_k^2)^{-1}\mathcal{H}_k^2 \mathbb{E}y_k\big]^T \mathcal{W}_k^2 \\ &\quad \times \big[v_k + (\mathcal{W}_k^2)^{-1}\mathcal{H}_k^2 \mathbb{E}y_k\big]\Big\}. \end{aligned}$$

Take

$$\begin{cases} v_k - \mathcal{I}_k^\theta \mathbb{E}v_k = -(\mathcal{W}_k^1)^{-1}\mathcal{H}_k^1(y_k - \mathcal{I}_k^\theta \mathbb{E}y_k) \equiv v_k^* - \mathcal{I}_k^\theta \mathbb{E}v_k^*, \\ \mathbb{E}v_k = -(\mathcal{W}_k^2)^{-1}\mathcal{H}_k^2 \mathbb{E}y_k \equiv \mathbb{E}v_k^*, \end{cases}$$

for $k \in \mathbb{T}$. Then, the minimum of $J(y_0, v; \theta_0)$ is achieved. Hence,

$$v_k^* = -(\mathcal{W}_k^1)^{-1}\mathcal{H}_k^1(y_k - \mathcal{I}_k^\theta \mathbb{E}y_k) - \mathcal{I}_k^\theta (\mathcal{W}_k^2)^{-1}\mathcal{H}_k^2 \mathbb{E}y_k. \tag{3.25}$$

As $(W_k^j)^{-1}H_k^j x_k I_{(\theta_k=j)} = (W_{\theta_k})^{-1}H_{\theta_k} x_k I_{(\theta_k=j)}$, we have

$$\begin{aligned} u_k^* = v_k^{*1} + \cdots + v_k^{*m} &= -(W_{\theta_k})^{-1}H_{\theta_k}x_k + \sum_{j=1}^m (W_k^j)^{-1} \\ &\quad \times H_k^j I_{(\theta_k=j)}\mathbb{E}y_k^j - \mathcal{I}_m \mathcal{I}_k^\theta (\mathcal{W}_k^2)^{-1}\mathcal{H}_k^2 \mathbb{E}y_k. \end{aligned}$$

This completes the proof. □

Remark 3.1. Note that $v_k^j = u_k I_{(\theta_k=j)}, j \in \mathcal{M}$. Then, $v_k^j, j \in \mathcal{M}$, cannot be designed independently. Therefore, we should show the well-posedness of the optimal control u^* (3.24) in the sense that from u^* we can get v^* (3.25). In fact,

$$\begin{aligned} u^* I_{(\theta_k=j)} &= I_{(\theta_k=j)}\Big[-(W_{\theta_k})^{-1}H_{\theta_k}x_k + \sum_{i=1}^m (W_k^i)^{-1} \\ &\quad \times H_k^i I_{(\theta_k=i)}\mathbb{E}y_k^i - \mathcal{I}_m \mathcal{I}_k^\theta (\mathcal{W}_k^2)^{-1}\mathcal{H}_k^2 \mathbb{E}y_k\Big] \\ &= -I_{(\theta_k=j)}\Big[(W_k^j)^{-1}H_k^j x_k - (W_k^j)^{-1}H_k^j \mathbb{E}y_k^j\Big] \\ &\quad - \Big[\mathcal{I}_k^\theta (\mathcal{W}_k^2)^{-1}\mathcal{H}_k^2 \mathbb{E}y_k\Big]_j, \end{aligned}$$

where $\Big[\mathcal{I}_k^\theta (\mathcal{W}_k^2)^{-1}\mathcal{H}_k^2 \mathbb{E}y_k\Big]_j$ denotes the jth line of p order of $\mathcal{I}_k^\theta (\mathcal{W}_k^2)^{-1}\mathcal{H}_k^2 \mathbb{E}y_k$. From (3.25) it follows that $u^* I_{(\theta_k=j)}$ equals v^{*j}. Therefore, u^* is well-posed.

Remark 3.2. Taking expectations for both sides of (3.25), we have

$$\begin{aligned} \mathbb{E}v_k^* &= -(\mathcal{W}_k^1)^{-1}\mathcal{H}_k^1\big(I - \mathbb{E}\mathcal{I}_k^\theta\big)\mathbb{E}y_k - \mathbb{E}\mathcal{I}_k^\theta (\mathcal{W}_k^2)^{-1}\mathcal{H}_k^2 \mathbb{E}y_k \\ &= -\big[(\mathcal{W}_k^1)^{-1}\mathcal{H}_k^1\big(I - \mathbb{P}_k\big) + \mathbb{P}_k(\mathcal{W}_k^2)^{-1}\mathcal{H}_k^2\big]\mathbb{E}y_k, \end{aligned}$$

where $\mathbb{P}_k$ is defined in (3.16). Under v^*, we have

$$\begin{aligned} &\mathbb{E}y_{k+1} \\ &= \big\{\bar{\mathcal{A}} - \bar{\mathcal{B}}\big[(\mathcal{W}_k^1)^{-1}\mathcal{H}_k^1\big(I - \mathbb{P}_k\big) + \mathbb{P}_k(\mathcal{W}_k^2)^{-1}\mathcal{H}_k^2\big]\big\}\,\mathbb{E}y_k, \end{aligned} \tag{3.26}$$

which is used in v^* and u^*. (3.26) is a deterministic linear system and the solution can be easily calculated.

Corollary 3.1. *When $\bar{Q}_{\theta_k} \equiv 0, \bar{R}_{\theta_k} \equiv 0, \bar{G}_{\theta_k} \equiv 0$ in (2.3), the corresponding optimal control is*

$$u_k^* = -(W_{\theta_k})^{-1}H_{\theta_k}x_{\theta_k}, \quad k \in \mathbb{T}. \tag{3.27}$$

Proof. In this case, $\mathcal{Q}_k^2, \mathcal{R}_k^2$ and $\mathcal{G}^2$ all equal 0. Hence, $\mathcal{S}_k \equiv 0$, $\mathcal{W}_k^2 \equiv \mathcal{W}_k^1$ and $\mathcal{H}_k^2 \equiv \mathcal{H}_k^1$, and hence, (3.24) reduces to (3.27). □

Remark 3.3. When $\bar{Q}_{\theta_k} \equiv 0, \bar{R}_{\theta_k} \equiv 0, \bar{G}_{\theta_N} \equiv 0$ in (2.3), Problem (MF-JLQ) reduces to

Problem (JLQ) :

$$\begin{cases} \text{Minimize } J(\zeta, u) = \sum_{k=0}^{N-1} \mathbb{E}\left[x_k^T Q_{\theta_k} x_k + u_k^T R_{\theta_k} u_k\right] \\ \qquad + \mathbb{E}\left(x_N^T G_{\theta_N} x_N\right), \\ \text{Subject to } x_{k+1} = \big[A_{\theta_k}x_k + B_{\theta_k}u_k\big] \\ \qquad + \big[C_{\theta_k}x_k + D_{\theta_k}u_k\big]w_k, \quad x_0 = \zeta, \ k \in \mathbb{T}. \end{cases}$$

This is a standard Markov jump LQ problem. Corollary 3.1 relives known result about Problem (JLQ). Hence, the decomposition technique of the state and the control provides an alternative method to deal with such problem.

4. A numerical example

This section studies a simple example of Problem (MF-LQ):

$$\begin{cases} \text{Minimize } J(\zeta, u; \theta_0) = \sum_{k=0}^{1} \mathbb{E}\left[q_{\theta_k}x_k^2 + \bar{q}_{\theta_k}(\mathbb{E}x_k)^2\right. \\ \qquad \left. + r_{\theta_k}u_k^2 + \bar{r}_{\theta_k}(\mathbb{E}u_k)^2\right] + g_{\theta_2}x_2^2 + \bar{g}_{\theta_2}(\mathbb{E}x_2)^2, \\ \text{Subject to } x_{k+1} = \big[a_{\theta_k}x_k + b_{\theta_k}u_k\big] + \big[c_{\theta_k}x_k + d_{\theta_k}u_k\big]w_k, \\ \qquad x_0 = \zeta \in \mathbb{R}^1, \ u \in \mathbb{R}^1. \end{cases}$$

Here, the Markov chain θ takes value in $\mathcal{M} = \{1, 2\}$ with transition probability matrix

$$\Lambda = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

whose (i, j)th element p_{ij} has the meaning

$$p_{ij} = P(\theta_{k+1} = j | \theta_k = i), \quad i, j = 1, 2, \ k = 0, 1, 2, \ldots.$$

The initial distribution of θ is $\nu = (\frac{1}{2}, \frac{1}{2})$. In this section, θ is assumed to be available to us. When $\theta_k = 1, k = 0, 1, 2, a_{\theta_k} = 1, b_{\theta_k} = 2, c_{\theta_k} = 3, d_{\theta_k} = 4, q_{\theta_k} = 1, \bar{q}_{\theta_k} = 2, r_{\theta_k} = 3, \bar{r}_{\theta_k} = 4, g_{\theta_2} = \bar{g}_{\theta_2} = 1$; when $\theta_k = 2, k = 0, 1, 2, a_{\theta_k} = 5, b_{\theta_k} = 6, c_{\theta_k} = 7, d_{\theta_k} = 8, q_{\theta_k} = 2, \bar{q}_{\theta_k} = 1, r_{\theta_k} = 1, \bar{r}_{\theta_k} = 2, g_{\theta_2} = \bar{g}_{\theta_2} = 2$.

It is easy to see that

$$(P(\theta_1 = 1), \quad P(\theta_1 = 2)) = \nu\Lambda = \left(\frac{3}{8}, \frac{5}{8}\right),$$

$$(P(\theta_2 = 1), \quad P(\theta_2 = 2)) = \nu\Lambda^2 = \left(\frac{13}{32}, \frac{19}{32}\right).$$

Using notations in Section 3, we have

$$\mathcal{A} = \begin{pmatrix} 1 & 5 \\ 1 & 5 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 2 & 6 \\ 2 & 6 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} 3 & 7 \\ 3 & 7 \end{pmatrix},$$

$$\mathcal{D} = \begin{pmatrix} 4 & 8 \\ 4 & 8 \end{pmatrix}, \quad \bar{\mathcal{A}} = \begin{pmatrix} \frac{1}{4} & \frac{5}{2} \\ \frac{3}{4} & \frac{5}{2} \end{pmatrix},$$

$$\bar{\mathcal{B}} = \begin{pmatrix} \frac{1}{2} & 3 \\ \frac{3}{2} & 3 \end{pmatrix}, \qquad \mathcal{Q}^1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\begin{cases} \bar{q}_0 = \mathbb{E}q_{\theta_0} = 2, \\ \bar{q}_1 = \mathbb{E}q_{\theta_1} = 2, \end{cases} \qquad \mathcal{Q}_0^2 = \mathcal{Q}_1^2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix},$$

$$\mathcal{R}^1 = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{cases} \bar{r}_0 = \mathbb{E}r_{\theta_0} = 3, \\ \bar{r}_1 = \mathbb{E}r_{\theta_1} = \frac{11}{4}, \end{cases},$$

$$\mathcal{R}_0^2 = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}, \qquad \mathcal{R}_1^2 = \begin{pmatrix} \frac{11}{4} & \frac{11}{4} \\ \frac{11}{4} & \frac{11}{4} \end{pmatrix},$$

$$\mathcal{G}^1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \bar{g} = \mathbb{E}g_{\theta_2} = \frac{51}{32}, \qquad \mathcal{G}^2 = \begin{pmatrix} \frac{51}{32} & \frac{51}{32} \\ \frac{51}{32} & \frac{51}{32} \end{pmatrix}.$$

From (3.18) and (3.20), we have

$$\mathcal{P}_1 = \begin{pmatrix} 2.7039 & 0 \\ 0 & 1.7947 \end{pmatrix}, \qquad \mathcal{P}_0 = \begin{pmatrix} 2.7729 & 0 \\ 0 & 1.8263 \end{pmatrix},$$

$$\mathcal{S}_1 = \begin{pmatrix} 3.1214 & 3.4297 \\ 3.4297 & 3.8975 \end{pmatrix}, \qquad \mathcal{S}_0 = \begin{pmatrix} 3.3155 & 3.5950 \\ 3.5950 & 4.0855 \end{pmatrix},$$

and

$$(\mathcal{W}_1^1)^{-1}\mathcal{H}_1^1 = \begin{pmatrix} 0.6447 & 0 \\ 0 & 0.8543 \end{pmatrix},$$

$$(\mathcal{W}_0^1)^{-1}\mathcal{H}_0^1 = \begin{pmatrix} 0.6517 & 0 \\ 0 & 0.8562 \end{pmatrix},$$

$$(\mathcal{W}_1^2)^{-1}\mathcal{H}_1^2 = \begin{pmatrix} 0.6111 & -0.0356 \\ -0.0142 & 0.8439 \end{pmatrix},$$

$$(\mathcal{W}_0^2)^{-1}\mathcal{H}_0^2 = \begin{pmatrix} 0.6088 & -0.0385 \\ -0.0139 & 0.8483 \end{pmatrix}.$$

Hence, the optimal control is

$$\begin{aligned} u_0^* = & -0.6517\Big(x_0 I_{(\theta_0=1)} - \mathbb{E}[x_0 I_{(\theta_0=1)}]\Big) \\ & - 0.8562\Big(x_0 I_{(\theta_0=2)} - \mathbb{E}[x_0 I_{(\theta_0=2)}]\Big) \\ & - I_{(\theta_0=1)}\Big(0.6088\mathbb{E}[x_0 I_{(\theta_0=1)}] - 0.0385\mathbb{E}[x_0 I_{(\theta_0=2)}]\Big) \\ & - I_{(\theta_0=2)}\Big(-0.0139\mathbb{E}[x_0 I_{(\theta_0=1)}] + 0.8483\mathbb{E}[x_0 I_{(\theta_0=2)}]\Big), \end{aligned} \quad (4.1)$$

$$\begin{aligned} u_1^* = & -0.6447\Big(x_1 I_{(\theta_1=1)} - \mathbb{E}[x_1 I_{(\theta_1=1)}]\Big) \\ & - 0.8543\Big(x_1 I_{(\theta_1=2)} - \mathbb{E}[x_1 I_{(\theta_1=2)}]\Big) \\ & - I_{(\theta_1=1)}\Big(0.6111\mathbb{E}[x_1 I_{(\theta_1=1)}] - 0.0356\mathbb{E}[x_1 I_{(\theta_1=2)}]\Big) \\ & - I_{(\theta_1=2)}\Big(-0.0142\mathbb{E}[x_1 I_{(\theta_1=1)}] + 0.8439\mathbb{E}[x_1 I_{(\theta_1=2)}]\Big). \end{aligned} \quad (4.2)$$

and the value function is

$$\begin{aligned} V(x_0, \theta_0) = & \, 2.7729\,\mathbb{E}\big[x_0^2 I_{(\theta_0=1)}\big] + 1.8263\,\mathbb{E}\big[x_0^2 I_{(\theta_0=2)}\big] \\ & + 3.3155\left(\mathbb{E}[x_0 I_{(\theta_0=1)}]\right)^2 + 2 \times 3.5950\,\mathbb{E}[x_0 I_{(\theta_0=1)}] \\ & \times \mathbb{E}[x_0 I_{(\theta_0=2)}] + 4.0855\left(\mathbb{E}[x_0 I_{(\theta_0=2)}]\right)^2. \end{aligned}$$

It is valuable to mention that the modes of the Markov chain are coupled in the optimal control. For instance, the terms $0.0385 I_{(\theta_0=1)}\mathbb{E}[x_0 I_{(\theta_0=2)}]$ and $0.0139 I_{(\theta_0=2)}\mathbb{E}[x_0 I_{(\theta_0=1)}]$ appear in (4.1). This is different from the known results about standard Markov jump LQ optimal control problems.

5. Conclusion

This paper considers the Markov jump mean-field LQ problem. Based on the modes of Markov chain and a decomposition technique of the state and the control, augmented state and control are introduced. Taking the decomposition and completing the square, an optimal control is constructed. It is shown that, under some appropriate conditions, there exists the unique optimal control, which can be explicitly presented via solutions to two generalized difference Riccati equations. For future researches, one can consider the Markov jump mean-field LQ problem with infinite horizon. Another topic is mean-field LQ problems with general random parameters. For this, a new yet fundamental methodology should be further developed. The backward stochastic difference equations might be the right one to tackle such a problem.

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References

[1] P.E. Caines, H.F. Chen, Optimal adaptive LQG control for systems with finite state process parameters, IEEE Trans. Automat. Control 30 (2) (1985) 185–185.
[2] P.E. Caines, J.F. Zhang, On the adaptive control for jump parameter systems via non-linear filtering, SIAM J. Control Optim. 33 (6) (1995) 1758–1777.
[3] M.D. Fragoso, O.L.V. Costa, A unified approach for stochastic and mean square stability of continuous-time linear systems with Markovian jumping parameters and additive disturbance, SIAM J. Control Optim. 44 (4) (2005) 1165–1191.
[4] Y. Ji, H.J. Chizeck, Controllability, observability and continuous-time Markovian jump linear quadratic control, IEEE Trans. Automat. Control 35 (1990) 777–788.
[5] X. Li, X.Y. Zhou, Indefinite stochastic LQ controls with Markovian jumps in a finite time horizon, Commun. Inf. Syst. 2 (2002) 265–282.
[6] X. Li, X.Y. Zhou, M. Ait Rami, Indefinite stochastic linear quadratic control with Markovian jumps in infinite time horizon, J. Global Optim. 27 (2003) 149–175.
[7] Z.B. Liu, Y.Z. Wang, H.T. Li, Two kinds of optimal controls for probabilistic mix-valued logical dynamic networks, Sci. China Inf. Sci. 57 (10) (2014) 052201.
[8] X. Mao, G. Yin, C. Yuan, Stabilization and destabilization of hybrid systems of stochastic differential equations, Automatica 43 (2007) 264–273.
[9] G. Yin, X.Y. Zhou, Markowitz mean–variance portfolio selection with regime switching: from discrete-time models to their continuous-time limits, IEEE Trans. Automat. Control 49 (2004) 349–360.
[10] O.L.V. Costa, M.D. Fragoso, R.P. Marques, Discrete-Time Markov Jump Linear Systems, Springer-Verlag, London, 2005.
[11] M. Mariton, Jump Linear Systems in Automatic Control, Marcel Decker, 1990.
[12] G. Yin, C. Zhu, Hybrid Switching Diffusions: Properties and Applications, Springer, New York, 2010.
[13] D. Ahdersson, B. Djehiche, A maximum principle for SDEs of Mean-field type, Appl. Math. Optim. 63 (3) (2011) 341–356.
[14] J. Li, Stochastic maximum principle in the mean-field controls, Automatica 48 (2012) 366–373.
[15] G. Wang, C. Zhang, W. Zhang, Stochastic maximum principle for mean-field type optimal control under partial information, IEEE Trans. Automat. Control 59 (2014) 522–528.
[16] J.M. Yong, A linear-quadratic optimal control problem for mean-field stochastic differential equations, SIAM J. Control Optim. 51 (2013) 2809–2838.
[17] R.J. Elliott, X. Li, Y.H. Ni, Discrete time mean-field stochastic linear-quadratic optimal control problems, Automatica 49 (11) (2013) 3222–3233.
[18] J. Huang, X. Li, J.M. Yong, A linear-quadratic optimal control problem for mean-field stochastic differential equations in infinite hrizon, arXiv:1208.5308.
[19] Y.H. Ni, R.J. Elliott, X. Li, Discrete-time mean-field stochastic linear-quadratic optimal controls. II—infinite horizon case, Automatica 57 (2015) 65–77.
[20] A. Bensoussan, J. Frehse, P. Yam, Mean Field Games and Mean Field Type Control Theory, in: SprigerBriefs in Mathematics, 2013.
[21] M.Y. Huang, P.E. Caines, R.P. Malhame, Individual and mass behaviour in large population stochastic wireless power control problems: centralized and Nash equilibrium solutions, in: Proceedings of the 42nd IEEE Conference on Decision and Control, Maui, Hawaii, December 2003, pp. 98–103.

[22] M.Y. Huang, P.E. Caines, R.P. Malhame, Large-population cost-coupled LQG problems with nonuniform agents: individual-mass behavior and decentralized ε-Nash equilibria, IEEE Trans. Automat. Control 52 (9) (2007) 1560–1571.
[23] M.Y. Huang, P.E. Caines, R.P. Malhame, Social optima in mean field LQG control: centralized and decentralized strategies, IEEE Trans. Automat. Control 57 (2010) 1736–1751.
[24] J.M. Lasry, P.L. Lions, Mean-field games, Jpn. J. Math. 2 (2007) 229–260.
[25] T. Li, J.F. Zhang, Asymptotically optimal decentralized control for large population stochastic multi-agent systems, IEEE Trans. Automat. Control 53 (2008) 1643–1660.
[26] H. Tembine, Q. Zhu, T. Basar, Risk-sensitive mean-field games, IEEE Trans. Automat. Control 59 (2014) 835–850.
[27] B.C. Wang, J.F. Zhang, Mean field games for large-popilation multi-agent systems with Markov jump parameters, SIAM J. Control Optim. 50 (2012) 2308–2334.
[28] A. Bensoussan, K. Sung, P. Yam, S. Yung, Linear-quadratic mean-field games, J. Optim. Theory Appl. (2015) 1–34.
[29] J.J.A. Hosking, A stochastic maximum principle for a stochastic differential game of a mean-ield type, Appl. Math. Optim. 66 (3) (2012) 415–454.